

The “Triptych” Function – Determining the Basins of Attraction of One-Dimensional Maps

I. C. Moreira

Instituto de Física – UFRJ, C. P. 68528, – Ilha do Fundão, 21945-970, Rio de Janeiro, Brazil

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We show that a modification of the path length method, introduced by R. Richter et al. [Z. Naturforsch. **49a**, 871 (1994)], gives us a method for finding basins of attraction of one-dimensional maps with multiple attractors.

In a recent paper [1], Richter et al. generalized the path length method [2] to dissipative systems and applied it to the logistic map. They found numerically the increments to the total path length of a trajectory (as a function of the initial condition and of the iteration number) and showed the emergence of a new fractal structure (the “triptych” principle). We show here that a modification of this method, with the introduction of an average increment on the path length given by the “triptych” function, can be used as a procedure for determining the basins of attraction of multiple attractors for one-dimensional maps.

The existence of multiple attractors is an important phenomenon for many nonlinear systems. Typical nonlinear dynamical systems may exhibit multiple co-existing (periodic or chaotic) attractors. In such cases the final state that is eventually reached depends on which basin of attraction the system is in at the initial instant of time. In analogy to the recent methods of controlling specific trajectories [3], we can use some procedures for controlling the final state to be reached by the system [4]; in this case we need to determine the basins of attraction of the different attractors. However, the determination of these basins is usually a difficult and time-consuming task that can be rarely done by analytical methods. Frequently the unique way of getting them is by using numerical calculations.

The k -order average increment of the path length of the trajectory, or average triptych function, will be defined as

$$T_k[F_N] = (1/N) \sum_{n=0}^N |X_{(n+1)k} - X_{nk}|.$$

This function measures, therefore, the average increment in the graph X_{i+k} versus X_i . If the trajectory is a periodic orbit with period k , the function $T_k[F_N]$ will tend to zero, when $N \gg 1$. If the trajectory has a different period, this function will give us the average increment at each k step. Therefore, by finding the value of T_k as a function of the initial condition (for $N \gg 1$) we can distinguish the basins of attraction of different attractors. For instance, if a map has two competing attractors, in the interval $[0, 1]$, the first one with period 1 and the second with period 2, the average triptych function of first-order will give zero in the basin of attraction of the first attractor and a value different from zero (the increment between the points on the attractor with period 2) in the basin of attraction of the second attractor. This fact gives us a quick method for identifying the different domains of attraction. We note that, as an interesting feature of this procedure, this function can distinguish also between two attractors with the same period, except if the increment, at each k step, is equal for the two attractors.

Let us consider two examples. The Tent-Bernoulli (TB)-map has the form

$$\begin{aligned} X_{i+1} &= 2aX_i, \quad \text{if } 0 \leq X_i \leq 1/2, \\ &= a[(4b-2)X_i + (2-3b)], \quad \text{if } 1/2 < X_i \leq 1. \end{aligned} \quad (1)$$

We take $b \in [0, 1]$ and we restricted, for our convenience, the domain of parameter a to be the interval $[0.9, 1]$. The parameter b is a relative measure of the gap of map (1) at the point $X = 1/2$ and a is a measure of the maximum height of map (1). This map is a discontinuous one-dimensional map which can be seen as a transition between the tent map and the Bernoulli shift: if $a = 1$ and $b = 0$ we get the tent map and, if $a = 1$ and $b = 1$, the Bernoulli shift. Several analytical results can be obtained for this map [5]. For

Reprint requests to Dr. I. C. Moreira, E-mail: ildeu@if.ufrj.br

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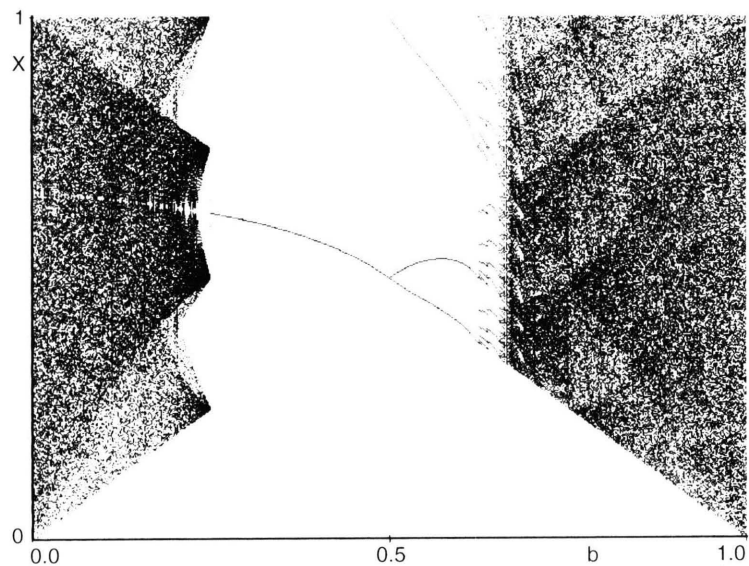


Fig. 1. X_∞ versus b , for the TB-map, with the initial condition $X_0 = 0.3$ and $a = 1$.

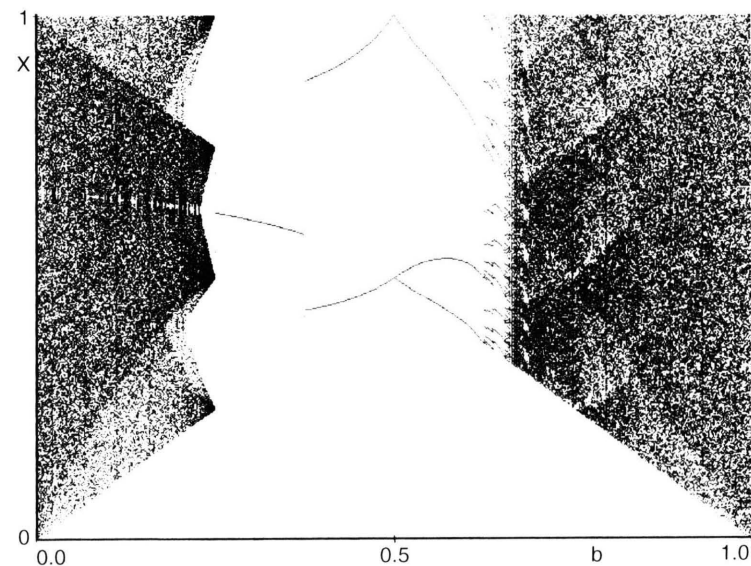


Fig. 2. X_∞ versus b , for the TB-map, with the initial condition $X_0 = 0.45$ and $a = 1$.

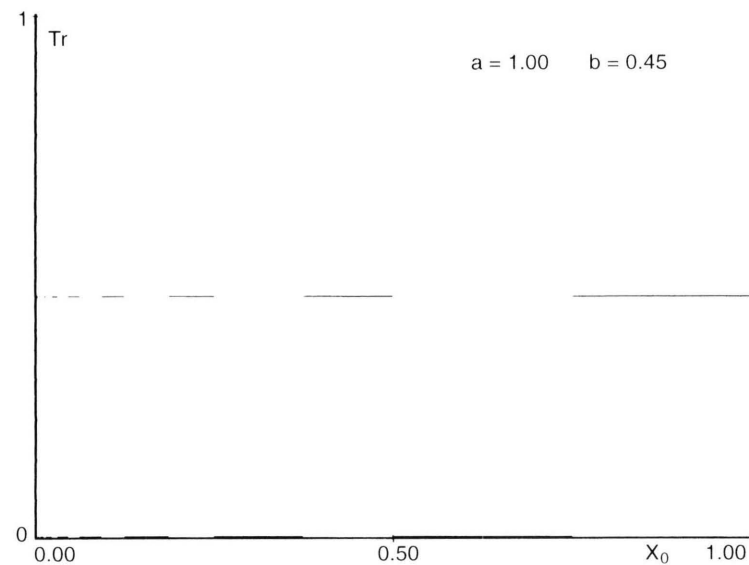


Fig. 3. Triptych function of first-order for the TB-map in the interval

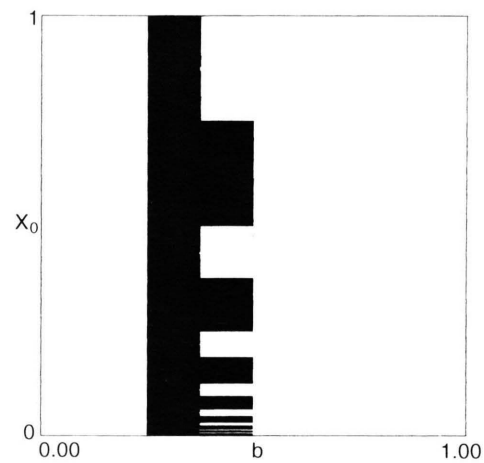


Fig. 4. Basin of attraction of the period 1 attractor (black) in the $b-X_0$

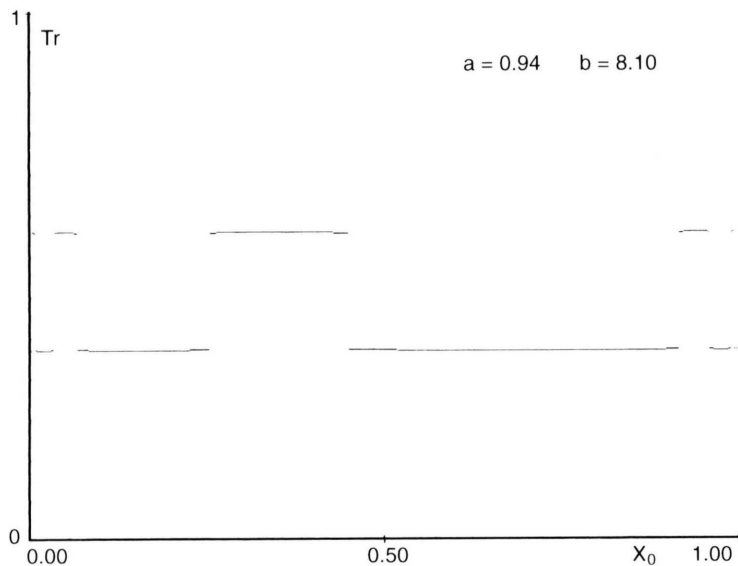


Fig. 5. The triptych function has different values for two attractors with the same period for the map (2).

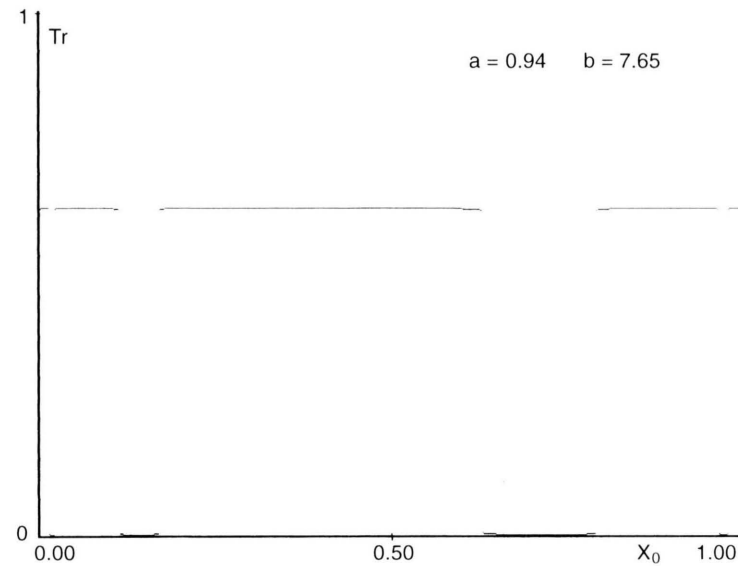


Fig. 6. Basins of attraction of a period 1 attractor and of another one with period 4 for the map (2).

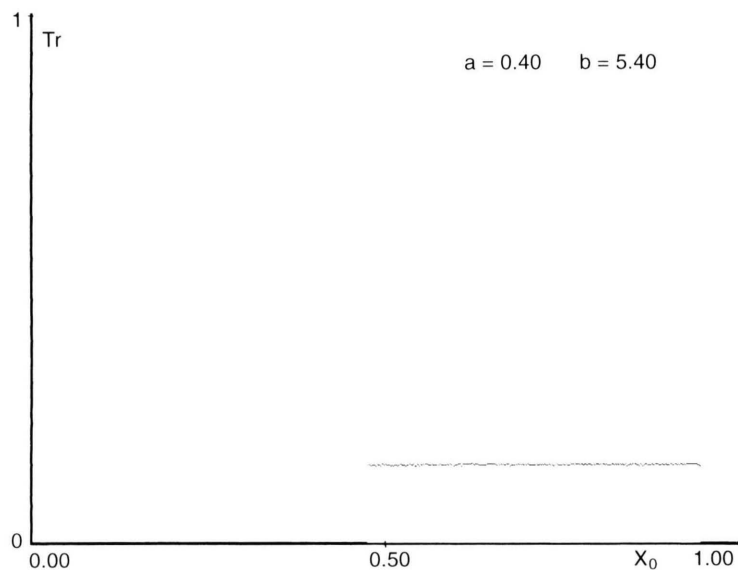


Fig. 7. Basins of attraction of a chaotic attractor and of a period 1 attractor for the map (2).

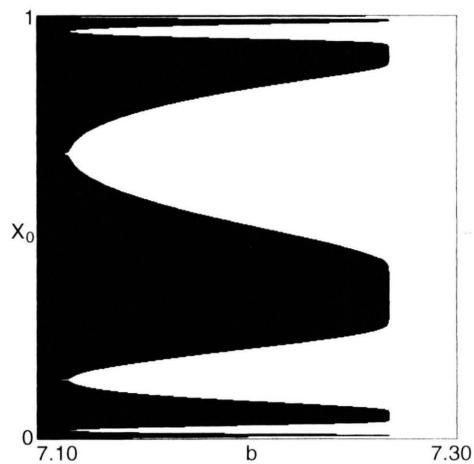


Fig. 8. Basins of attraction of a period 1 attractor (white) and of a period 2 attractor (black) for the map (2), with $a = 0.9$.

a fixed a , if b is varied from 0 to 1 the behavior of the map changes between chaotic and regular. We are interested here in the domains of a and b where two different attractors coexist. The analysis of the first and second return maps shows that, in the interval $b \in [0, 1/2]$, one attractor with period 1 (P-I) exists if $1/2 - (1/4a) < b < 1 - (1/2a)$ and one attractor of period 2 (P-II) exists if $b > 1/2 - (1/8a^2)$. Therefore, the coexistence of these attractors occurs if $1/2 - (1/8a^2) < b < 1 - (1/2a)$ [see Figs. 1 and 2].

The basins of attraction of these two periodic attractors, for $b = 0.45$ and $a = 1.0$, obtained by using the first-order triptych function are represented in Figure 3. The value of the triptych function for the basin of attraction of P-I is zero. We note that, near $X = 0$, the basins of attraction are interwoven. In Fig. 4 the entire basin of attraction of P-I in the $b - X_0$ space is shown. The number of iterations is $N = 2000$ for all figures.

The method was applied also for determining the basins of attraction of a family of continuous one-dimensional maps with the form [6]

$$X_{i+1} = bX_i + (-7b + 135a/4)X_i^2 + (15b - 189a/2)X_i^3 + (243a/4 - 9b)X_i^4. \quad (2)$$

This map has in the interval $[0, 1]$, several coexisting (periodic or chaotic) attractors with blended basins of attraction for several values of the parameters a and b . We have constructed this map by violating the conditions of Singer's theorem [7] for the existence of only one attractor for continuous one-dimensional maps and, by imposing some additional conditions for getting a simple map, with two parameters, in the interval $[0, 1]$. For instance, Fig. 5 shows the basin of attraction for two attractors both with period 2, if $a = 0.94$ and $b = 8.10$. In Fig. 6, the triptych function permits us to determine the basin of an attractor with period 4 and the basin of another attractor with period 1. In Fig. 7, where we used the triptych function of first-order, we can see the basins of attraction of a chaotic attractor and of a period 1 attractor. Figure 8 shows the entire basin of attraction of two attractors in the interval $b \in [7.1, 7.3]$, if $a = 0.9$.

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